A meshless based method for solution of integral equations: Improving the error analysis

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Abstract

This draft concerns the error analysis of a collocation method based on the moving least squares (MLS) approximation for integral equations, which improves the results of [2] in the analysis part. This is mainly a translation from Persian of some parts of Chapter 2 of the author's PhD thesis in 2011.

1 Introduction

In [2] a meshless method based on the *moving least squares* (MLS) was applied for integral equations of the second kind, and an error analysis was presented for Fredholm integral equations. Here a more interesting presentation of the MLS approximation and its error estimation are reported, and the analysis of the MLS collocation method for Fredholm integral equations of the second kind is revised. The analysis is mainly based on the excellent book [1].

2 MLS approximation

Let $\Omega \subset \mathbb{R}^d$, for positive integer d, be a nonempty and bounded set. Assume,

$$X = \{x_1, x_2, \dots, x_N\} \subset \Omega,$$

is a set containing N scattered points. The *fill distance* of X is defined to be

$$h_{X,\Omega} = \sup_{x\in\Omega} \min_{1\leqslant j\leqslant N} \|x - x_j\|_2,$$

and the *separation distance* is defined by

$$q_X = \frac{1}{2} \min_{i \neq j} \|x_i - x_j\|_2.$$

A set X of data sites is said to be *quasi-uniform* with respect to a constant $c_{\mathrm{qu}} > 0$ if

$$q_X \leqslant h_{X,\Omega} \leqslant c_{\mathrm{qu}} q_X. \tag{1}$$

Henceforth, we use $\mathbb{P}_m^d$, for $m \in \mathbb{N}_0 = \{n \in \mathbb{Z}, n \geqslant 0\}$, as the space of d-variable polynomials of degree at most m of dimension $Q := \binom{m+d}{d}$. A basis for this space is denoted by $\{p_1, \ldots, p_Q\}$ or $\{p_\alpha\}_{0 \leqslant |\alpha| \leqslant m}$.

A set $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ with $N \geqslant Q$ is called $\mathbb{P}_m^d$-unisolvent if the zero polynomial is the only polynomial from $\mathbb{P}_m^d$ that vanishes on X.

The MLS provides an approximation $s_{u,X}$ of u in terms of values $u(x_j)$ at centers x_j by

$$u(x) \approx s_{u,X}(x) = \sum_{j=1}^{N} \phi_j(x) u(x_j), \quad x \in \Omega, \tag{2}$$

where ϕ_j are *MLS shape functions* given by

$$\phi_j(x) = w(x, x_j) \sum_{k=1}^{Q} \lambda_k(x) p_k(x_j),$$

where the influence of the centers is governed by a weight function $w_j(x) = w(x, x_j)$, which vanishes for arguments $x, x_j \in \Omega$ with $\|x - x_j\|_2$ greater than a certain threshold, say δ. Thus we can define $w_j(x) = K((x - x_j)/\delta)$ where $K : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative function with support in the unit ball $B(0, 1)$. Coefficients $\lambda_k(x)$ are the unique solution of

$$\sum_{k=1}^{Q} \lambda_k(x) \sum_{j \in J(x)} w_j(x) p_k(x_j) p_\ell(x_j) = p_\ell(x), \quad 0 \leqslant \ell \leqslant Q,$$

where $J(x) = \{j : \|x - x_j\|_2 \leqslant \delta\}$ is the family of indices of points in the support of the weight function. In vector form

$$\boldsymbol{\phi}(x) = W(x) P^T (P W(x) P^T)^{-1} \boldsymbol{p}(x),$$

where $W(x)$ is the diagonal matrix carrying the weights $w_j(x)$ on its diagonal, P is a $Q \times \# J(x)$ matrix of values $p_k(x_j)$, $j \in J(x)$, $1 \leqslant k \leqslant Q$, and $\boldsymbol{p} = (p_1, \ldots, p_Q)^T$. In the MLS, one finds the best approximation to u at point x, out of $\mathbb{P}_m^d$ with respect to a discrete ℓ^2 norm induced by a *moving* inner product, where the corresponding

weight function depends not only on points x_j but also on the evaluation point x in question. Note that, if for every $x \in \Omega$ the set $\{x_j : j \in J(x)\}$ is $\mathbb{P}_m^d$-unisolvent then $A(x) = PW(x)P^T$ is a symmetric positive definite matrix. More details can be found in Chapter 4 of [4]. In what follows we will assume that K is nonnegative and continuous on $\mathbb{R}^d$ and positive on the ball $B(0, 1/2)$. In many applications we can assume that

$$K(x) = \varphi(\|x\|_2), \quad x \in \mathbb{R}^d,$$

meaning that K is a radial function. Here $\varphi : [0, \infty) \to \mathbb{R}$ is positive on $[0, 1/2]$, supported in $[0, 1]$ and its even extension is nonnegative and continuous on $\mathbb{R}$. If we assume that $K \in C^k(\mathbb{R}^d)$ then $\phi_j \in C^n(\Omega)$ where $n = \min\{k, m\}$. This implies that $s_{u,X} \in C^n(\Omega)$.

It is well-known that [4] if $X = \{x_1, \ldots, x_N\}$ is a quasi-uniform set in $\Omega \subset \mathbb{R}^d$, where Ω is a compact set and satisfies an interior cone condition, then the MLS shape functions $\{\phi_j\}$ provide a *stable local polynomial reproduction* of degree $m \in \mathbb{N}_0$ on Ω, i.e. there exist constants $h_0, C_1, C_2 > 0$ independent of X such that for every $x \in \Omega$

1. $\sum_{j=1}^N \phi_j(x)p(x_j) = p(x)$, $\forall p \in \mathbb{P}_m^d$,
2. $\sum_{j=1}^N |\phi_j(x)| \leqslant C_1$,
3. $\phi_j(x) = 0$ if $\|x - x_j\|_2 > \delta = 2C_2 h_{X,\Omega}$,

for all X with $h_{X,\Omega} \leqslant h_0$.

Note that, a set $\Omega \subset \mathbb{R}^d$ is said to satisfy an interior cone condition if there exist an angle $\theta \in (0, \pi/2)$ and a radius $r > 0$ such that for every $x \in \Omega$ a unit vector $\xi(x)$ exists such that the cone

$$C(x, \xi, \theta, r) := \left\{x + ty : y \in \mathbb{R}^d, \|y\|_2 = 1, y^T\xi \geqslant \cos\theta, t \in [0, r]\right\}$$

is contained in Ω.

The following theorem shows that the MLS approximation converges uniformly for continuous functions on compact domain Ω.

Theorem 2.1. *Suppose that $\Omega \subset \mathbb{R}^d$ is compact and satisfies an interior cone condition. The MLS approximation $s_{u,X}$ converges uniformly for all continuous function u, as $h_{X,\Omega}$ goes to zero for quasi-uniform sets X.*

Proof. For a fixed $x \in \Omega$, suppose that p_0 is the constant polynomial with $p_0(x) = u(x)$. The conditions of Theorem ensure that the MLS shape functions provide a

stable local polynomial reproduction. Thus we can write

$$\begin{aligned}
|u(x) - s_{u,X}(x)| &= \Big| p_0(x) - \sum_{j=1}^{N} \phi_j(x) u(x_j) \Big| \\
&= \Big| \sum_{j=1}^{N} \phi_j(x) \big(p_0(x_j) - u(x_j) \big) \Big| \\
&\leq \sum_{j=1}^{N} |\phi_j(x)| \big| p_0(x_j) - u(x_j) \big| \\
&\leq C_1 \|u - p_0\|_{\infty, B(x,\delta)\cap\Omega} \\
&= C_1 \max_{y \in B(x,\delta)\cap\Omega} |u(y) - u(x)| \\
&\leq C_1 \omega(u, \delta),
\end{aligned}$$

where $\omega(u,\delta)$ is the modulus of continuity of u. The compactness of Ω and $\delta = ch_{X,\Omega}$ give the uniform convergence. $\square$

Finally, the following error estimation can be proved for smoother functions. Note that a domain with a Lipschitz boundary, automatically satisfies an interior cone condition. The reader is referred to [3] for proof.

Theorem 2.2. *Suppose that $\Omega \subset \mathbb{R}^d$ is a compact set with a Lipschitz boundary. Let m be a positive integer. If $u \in W_\infty^{m+1}(\Omega)$, there exist constants $C > 0$ and $h_0 > 0$ such that for all $X = \{x_1, \ldots, x_N\} \subset \Omega$ with $h_{X,\Omega} \leqslant h_0$ which are quasi-uniform with the same c_{qu} in (1), the estimate*

$$\|u - s_{u,X}\|_{L_\infty(\Omega)} \leqslant C h_{X,\Omega}^{m+1} \|u\|_{W_\infty^{m+1}(\Omega)} \tag{3}$$

holds.

In numerical implementation, for computing the MLS approximation at a sample point $\widehat{x} \in \Omega$, the *shifted* and *scaled* polynomial basis functions

$$\left\{ \frac{(x - \widehat{x})^\alpha}{h_{X,\Omega}^{|\alpha|}} \right\}_{0 \leq \alpha \leq m}$$

are used as a basis for $\mathbb{P}_m^d$. In fact we change the basis functions as the evaluation point is changed. This leads to a more stable algorithm. For example, the use of shifted and scaled basis functions overcomes the instability of the reported results in Tables 1, 4 and 6 of [2, Section 6] for quadratic basis functions.

3 The MLS collocation method

A Fredholm integral equation of the second kind can be written as

$$\lambda u(x) + \int_\Omega \kappa(x,s)u(s)ds = f(x), \quad x \in \Omega \subset \mathbb{R}^d, \tag{4}$$

where u is an unknown function, λ is a real parameter, Ω is a compact domain in $\mathbb{R}^d$, f is a given continuous right-hand side function, and κ is a given continuous kernel in $\Omega \times \Omega$. The above integral equation can be written in the following abstract form,

$$(\lambda - \mathcal{F})u = f,$$

where

$$\mathcal{F}u = \int_\Omega \kappa(x,s)u(s)ds.$$

We consider a set of *trial points* $X = \{x_1, x_2, \ldots, x_N\} \subset \Omega$ with fill distance $h_{X,\Omega}$. Regarding the previous section, we assume that X is a quasi-uniform set and admits a well-defined MLS approximation. Suppose that $\phi_1, \ldots, \phi_N$ are the MLS shape functions constructed by polynomial space $\mathbb{P}_m^d$ and weight function $K \in C^k(\mathbb{R}^d)$, $k \in \mathbb{N}_0$. Define

$$V_N := \operatorname{span}\{\phi_1, \ldots, \phi_N\},$$

as a finite dimensional subspace of $C(\Omega)$. According to (2), the MLS approximation $\widehat{u} := s_{u,X}$ of u is

$$u \approx \widehat{u} = \sum_{j=1}^N \phi_j u(x_j) \in V_N.$$

Moreover, we define a projection operator $\mathcal{P}_N : C(\Omega) \mapsto V_N$ which interpolates any continuous function into V_N on *test points* $Y = \{y_1, \ldots, y_M\} \subset \Omega$. More precisely, for all $u \in C(\Omega)$ we define

$$\mathcal{P}_N u := \sum_{j=1}^N \phi_j c_j, \text{ with } \mathcal{P}_N u(y_i) = u(y_i),\ 1 \le i \le M.$$

In what follows, we let $M = N$ and we assume that, the distribution of both sets of test and trial points are well enough to ensure the non-singularity of $\Phi_N = \big(\phi_j(y_k)\big)_{i,j=1}^N$. If it happens then $\mathcal{P}_N$ is well-defined. Since $\widehat{u} \in V_N$, we simply have $\mathcal{P}_N \widehat{u} = \widehat{u}$. Replacing u by $\widehat{u}$ in (4) we get

$$\sum_{j=1}^N \left[\lambda\phi_j(x) + \int_\Omega \kappa(x,s)\phi_j(s)ds\right] u(x_j) = f(x) + r(x),$$

where $r(x)$ is the reminder. In the *collocation* method we assume that the reminder is vanished at test points Y, i.e.

$$\mathcal{P}_N r = 0,$$

which leads to

$$\sum_{j=1}^{N} \left[\lambda \phi_j(y_i) + \int_\Omega \kappa(y_i, s)\phi_j(s)ds \right] u(x_j) = f(y_i), \quad 1 \le i \le N,$$

or in an abstract form

$$\mathcal{P}_N(\lambda - \mathcal{F})\widehat{u} = \mathcal{P}_N f.$$

According to the property $\mathcal{P}_N \widehat{u} = \widehat{u}$, we have

$$(\lambda - \mathcal{P}_N \mathcal{F})\widehat{u} = \mathcal{P}_N f.$$

The involved integral can be treated by a numerical quadrature of the form

$$\int_\Omega g(s)\, ds \approx \sum_{k=1}^{Q_N} g(\tau_k)\omega_k, \quad g \in C(\Omega), \tag{5}$$

where $\{\tau_k\}$ and $\{\omega_k\}$, for $1 \le k \le Q_N$ are integration points and weights, respectively. We assume that for all $g \in C(\Omega)$ the quadrature converges to the exact value of integral as Q_N increases. Now we define

$$\mathcal{F}_N u(x) := \sum_{k=1}^{Q_N} \kappa(x, \tau_k) u(\tau_k)\omega_k, \quad x \in \Omega,\ u \in C(\Omega). \tag{6}$$

If we replace $\mathcal{F}\widehat{u}$ by $\mathcal{F}_N \widehat{u}$, we will get

$$\sum_{j=1}^{N} \left[\lambda \phi_j(y_i) + \sum_{k=1}^{Q_N} \kappa(y_i, \tau_k)\phi_j(\tau_k)\omega_k \right] \widetilde{u}_j = f(y_i), \quad 1 \le i \le N, \tag{7}$$

where $\widetilde{u}_j$ are the approximation values of $u(x_j)$. Solving the linear system of equations (7) gives the values $\widetilde{u}_j$, $j = 1, \ldots, N$, and finally one can approximate

$$u(x) \approx u_N(x) = \sum_{j=1}^{N} \phi_j(x)\widetilde{u}_j,$$

for any $x \in \Omega$. The abstract form of equation (7) is

$$\mathcal{P}_N(\lambda - \mathcal{F}_N)u_N = \mathcal{P}_N f.$$

Since $u_N \in V_N$, we have $\mathcal{P}_N u_N = u_N$ and the above equation can be rewritten as

$$(\lambda - \mathcal{P}_N \mathcal{F}_N) u_N = \mathcal{P}_N f \tag{8}$$

which shows that the scheme is a *discrete collocation method* [1]. Consequently an *iterated discrete collocation* solution can be obtained. For this purpose we set

$$v_N(x) = \frac{1}{\lambda}[f(x) + \mathcal{F}_N u_N(x)], \quad \forall x \in \Omega, \tag{9}$$

and by applying the operator $\mathcal{P}_N$ on both sides of (9), and using the relation (8) we simply have

$$\mathcal{P}_N v_N = u_N.$$

Thus we conclude

$$(\lambda - \mathcal{F}_N \mathcal{P}_N) v_N = f. \tag{10}$$

Equations (8) and (10) will be referred in the next section when we will try to give the error bounds for $u - u_N$ and $u - v_N$.

Usually and in this paper the case $M = N$ is assumed which leads to a square final linear system. In addition we can assume that $X = Y$. The case $M > N$ is called *oversampling* which may help if there is a problem with solvability.

4 Error Analysis

As we discussed in the previous section, the method is a discrete collocation, and the solvability of the integral equation (4) and some insights on integration operators $\mathcal{F}_N$ and projections $\mathcal{P}_N$ are required to obtain the final error bound. Moreover, an error bound for the MLS approximation should be invoked.

According to (6) we define

$$\|\mathcal{F}_N\| := \max_{x\in\Omega} \sum_{k=1}^{K_N} |\omega_k \kappa(x, \tau_k)|.$$

A direction which makes the analysis possible is to seek for characteristic properties of operators $\mathcal{F}_N$ which imply

$$\|(\mathcal{F} - \mathcal{F}_N)\mathcal{F}\| \to 0, \quad \|(\mathcal{F} - \mathcal{F}_N)\mathcal{F}_N\| \to 0, \text{ as } N \to \infty. \tag{11}$$

For this, we assume that $\{\mathcal{F}_N,\ N \geq 1\}$ possesses the following properties:

1. $\mathcal{X}$ is a Banach space, and $\mathcal{F}$ and $\mathcal{F}_N$, for $N \geq 1$, are linear operator on $\mathcal{X}$ into $\mathcal{X}$.

2. $\mathcal{F}_N u \to \mathcal{F} u$ as $N \to \infty$, for all $u \in \mathcal{X}$.

3. The set $\{\mathcal{F}_N,\, N \geq 1\}$ is collectively compact which means that $\{\mathcal{F}_N u,\, N \geq 1,\, \|u\| \leq 1\}$ has a compact closure in $\mathcal{X}$.

Then $\{\mathcal{F}_N\}$ is said to be a *collectively compact family of pointwise convergent operators.* According to [1, Lemma 4.1.2], if $\{\mathcal{F}_N,\, N \geq 1\}$ is a collectively compact family of pointwise convergent operators, then (11) is satisfied. Finally, [1, Theorem 4.1.1] paves the way for finding the final error bound.

Theorem 4.1. *Let $\mathcal{X}$ be a Banach space, let $\mathcal{S}$ and $\mathcal{T}$ be bounded operators on $\mathcal{X}$ to $\mathcal{X}$ and let $\mathcal{S}$ be compact. For given $\lambda \neq 0$, assume $\lambda - \mathcal{T}:\ \mathcal{X} \xrightarrow[onto]{1-1} \mathcal{X}$, which implies $(\lambda - \mathcal{T})^{-1}$ exists as a bounded operator on $\mathcal{X}$ to $\mathcal{X}$. Finally assume*

$$\|(\mathcal{T} - \mathcal{S})\mathcal{S}\| < \frac{|\lambda|}{\|(\lambda - \mathcal{T})^{-1}\|}, \tag{12}$$

then $(\lambda - \mathcal{S})^{-1}$ exists and it is a bounded operator from $\mathcal{X}$ to $\mathcal{X}$. In fact, we have

$$\|(\lambda - \mathcal{S})^{-1}\| \leq \frac{1 + \|(\lambda - \mathcal{T})^{-1}\|\|\mathcal{S}\|}{|\lambda| - \|(\lambda - \mathcal{T})^{-1}\|\|(\mathcal{T} - \mathcal{S})\mathcal{S}\|}. \tag{13}$$

If $(\lambda - \mathcal{T})w = f$ and $(\lambda - \mathcal{S})z = f$, then

$$\|w - z\| \leq \|(\lambda - \mathcal{S})^{-1}\|\|\mathcal{T}w - \mathcal{S}w\|. \tag{14}$$

Now we go back to equations (8) and (10). In section 3.4 of [1] it is proved that the existence of the inverse operators $(\lambda - \mathcal{F}_N\mathcal{P}_N)^{-1}$ and $(\lambda - \mathcal{P}_N\mathcal{F}_N)^{-1}$ are related to each other. If $(\lambda - \mathcal{P}_N\mathcal{F}_N)^{-1}$ exists, then so does $(\lambda - \mathcal{F}_N\mathcal{P}_N)^{-1}$ and

$$(\lambda - \mathcal{F}_N\mathcal{P}_N)^{-1} = \frac{1}{\lambda}[I + \mathcal{F}_N(\lambda - \mathcal{P}_N\mathcal{F}_N)^{-1}\mathcal{P}_N].$$

Conversely, if $(\lambda - \mathcal{F}_N\mathcal{P}_N)^{-1}$ exists, then so does $(\lambda - \mathcal{P}_N\mathcal{F}_N)^{-1}$ and

$$(\lambda - \mathcal{P}_N\mathcal{F}_N)^{-1} = \frac{1}{\lambda}[I + \mathcal{P}_N(\lambda - \mathcal{F}_N\mathcal{P}_N)^{-1}\mathcal{F}_N].$$

By combining these, we also have

$$(\lambda - \mathcal{P}_N\mathcal{F}_N)^{-1}\mathcal{P}_N = \mathcal{P}_N(\lambda - \mathcal{F}_N\mathcal{P}_N)^{-1}.$$

We can choose to show the existence of either $(\lambda - \mathcal{F}_N\mathcal{P}_N)^{-1}$ or $(\lambda - \mathcal{P}_N\mathcal{F}_N)^{-1}$ whichever is the more convenient, and the existence of the other inverse will follow immediately.

To use the results of Theorem 4.1 for schemes (8) and (10), we should first prove that $\{\mathcal{F}_N\mathcal{P}_N,\ N \geq 1\}$ is a "collectively compact family of pointwise convergent operators". To this aim, we need a uniform bound for $\|\mathcal{P}_N\|$.

Lemma 4.2. *Assume that $\phi_1,\dots,\phi_N$ are the MLS shape functions on the quasi uniform set $X=\{x_1,\dots,x_N\}$ with fill distance $h_{X,\Omega}$ on a compact domain Ω which satisfies an interior cone condition. If $\|\Phi_N^{-1}\|_\infty=\mathcal{O}(1)$ independent of N (or $h_{X,\Omega}$), then there exists a constant c_P independent of N such that $\|\mathcal{P}_N\|\le c_P$, and $\mathcal{P}_N u\to u$ uniformly for all $u\in C(\Omega)$.*

Proof. First

$$\mathcal{P}_N u(x)=\sum_{j=1}^N \phi_j(x)c_j,\quad \text{and}\quad \mathcal{P}_N u(y_i)=u(y_i),\ 1\le i\le N,$$

give $\boldsymbol{c}=\Phi_N^{-1}\boldsymbol{u}$. On the other hand we have

$$\|\mathcal{P}_N u\|_\infty\le\|\boldsymbol{c}\|_\infty\max_{x\in\Omega}\sum_{j=1}^N|\phi_j(x)|\le C_1\|\Phi_N^{-1}\|_\infty\|u\|_\infty.$$

The last inequality is satisfied because of the L_1 stability of the MLS shape functions (the second property of a stable local polynomial reproduction). Thus we can write

$$\|\mathcal{P}_N\|=\sup_{u\in C(\Omega)}\frac{\|\mathcal{P}_N u\|_\infty}{\|u\|_\infty}\le C_1\|\Phi_N^{-1}\|_\infty,$$

which leads to

$$c_P:=\sup_N\|\mathcal{P}_N\|<\infty. \tag{15}$$

Finally, if $\widehat{u}$ is the MLS approximation of u on X then

$$\begin{aligned}\|\mathcal{P}_N u-u\|_\infty&\le\|\mathcal{P}_N u-\mathcal{P}_N\widehat{u}\|_\infty+\|u-\widehat{u}\|_\infty\\&\le(1+c_P)\|u-\widehat{u}\|_\infty\\&\le C(1+c_P)\omega(u,h_{X,\Omega})\end{aligned}$$

In the first inequality we have used $\mathcal{P}_N\widehat{u}=\widehat{u}$, and in the last one we have applied Theorem 2.1. Since the points are quasi uniform, $N\to\infty$ implies $h_{X,\Omega}\to 0$ and $\mathcal{P}_N u\to u$, uniformly. $\square$

Remark 4.3. *Experiments show that $\|\Phi_N^\dagger\|_\infty$ is of order 1 independent of the fill distance $h_{X,\Omega}$ even if $M=N$. But it remains to prove this assertion, theoretically.*

In the following lemma we prove that under some conditions $\{\mathcal{F}_N\mathcal{P}_N,\ N\ge 1\}$ is a collectively compact family of pointwise convergent operators.

Lemma 4.4. *Assume that $\{\mathcal{F}_N,\ N\ge1\}$ is a collectively compact family of pointwise convergent operators on $\mathcal{X}=C(\Omega)$. Then $\{\mathcal{F}_N\mathcal{P}_N,\ N\ge1\}$ is a collectively compact family of pointwise convergent operators on $\mathcal{X}$.*

Proof. From (15) we have $c_P \equiv \sup \|\mathcal{P}_N\| < \infty$. The pointwise convergence of $\{\mathcal{F}_N\}$ implies that $c_F \equiv \sup \|\mathcal{F}_N\| < \infty$. Together, these imply the uniform boundedness of $\{\mathcal{F}_N\mathcal{P}_N\}$ with a bound of $c_P c_F$. For the pointwise convergence on $C(\Omega)$ we have

$$\begin{aligned}\|\mathcal{F}u - \mathcal{F}_N\mathcal{P}_N u\|_\infty &\le \|\mathcal{F}u - \mathcal{F}_N u\|_\infty + \|\mathcal{F}_N(u - \mathcal{P}_N u)\|_\infty \\ &\le \|\mathcal{F}u - \mathcal{F}_N u\|_\infty + c_F\|u - \mathcal{P}_N u\|_\infty,\end{aligned}$$

and the convergence now follows from that of $\{\mathcal{F}_N u\}$ and $\{\mathcal{P}_N u\}$. To prove the collective compactness of $\{\mathcal{F}_N\mathcal{P}_N\}$ we must show that

$$\mathcal{K} = \{\mathcal{F}_N\mathcal{P}_N u \,:\, N \ge 1,\ \|u\|_\infty \le 1\}$$

has a compact closure in $C(\Omega)$. From (15) we have

$$\mathcal{K} \subset \{\mathcal{F}_N u \,:\, N \ge 1,\ \|u\|_\infty \le c_P\}$$

which proves the assertion because $\{\mathcal{F}_N\}$ is collectively compact. □

Now (11) is satisfied by replacing $\mathcal{F}_N$ by $\mathcal{F}_N\mathcal{P}_N$, and we can apply Theorem 4.1.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^d$ be a compact set with a Lipschitz boundary, and the quasi uniform set $X = \{x_1, \ldots, x_N\} \subset \Omega$ be a set of trial points with fill distance $h_{X,\Omega}$. Let $\{\mathcal{P}_N\}$ be a family of interpolant operators from $C(\Omega)$ to $V_N = \mathrm{span}\{\phi_1, \ldots, \phi_N\}$ on test points $Y = \{y_1, \ldots, y_N\} \subset \Omega$, where ϕ_j are the MLS shape functions based on X and polynomial space $\mathbb{P}_m^d$. Assume that the distribution of points is well enough to ensure the non-singularity of Φ_N, and $\|\Phi_N^{-1}\|_\infty = \mathcal{O}(1)$. Further, assume that $\{\mathcal{F}_N\}$ in (6) is a collectively compact family of pointwise convergent operators on $C(\Omega)$. Finally, assume that $(\lambda - \mathcal{F})u = f$ is uniquely solvable for all $f \in C(\Omega)$. Then for all sufficiently large N, say $N \ge N_0$, the operator $(\lambda - \mathcal{F}_N\mathcal{P}_N)^{-1}$ exists and it is uniformly bounded. In addition, for the iterative solution v_N for equation $(\lambda - \mathcal{F}_N\mathcal{P}_N)v_N = f$ we have*

$$\|u - v_N\|_\infty \le c_I\big\{\|\mathcal{F}_N u - \mathcal{F}u\|_\infty + c_F(1 + c_P)Ch_{X,\Omega}^{m+1}\|u\|_{W_\infty^{m+1}(\Omega)}\big\}, \tag{16}$$

provided that $u \in W_\infty^{m+1}(\Omega)$, and for the discrete collocation solution u_N of equation $(\lambda - \mathcal{P}_N\mathcal{F}_N)u_N = \mathcal{P}_N f$ we have

$$\|u - u_N\|_\infty \le c_I\big\{c_P\|\mathcal{F}_N u - \mathcal{F}u\|_\infty + (1 + c_P c_F)c_F(1 + c_P)Ch_{X,\Omega}^{m+1}\|u\|_{W_\infty^{m+1}(\Omega)}\big\}, \tag{17}$$

where $c_I < \infty$ is a bound for $(\lambda - \mathcal{F}_N\mathcal{P}_N)^{-1}$.

Proof. According to the assumptions and using Lemma 4.4 we conclude that $\{\mathcal{F}_N\mathcal{P}_N\}$ is a collectively compact family of pointwise convergent operators. By Lemma **??** and the discussions after that, we have

$$\|(\mathcal{F}-\mathcal{F}_N\mathcal{P}_N)\mathcal{F}_N\mathcal{P}_N\| \to 0.$$

Thus, (12) is satisfied for $N \geq N_0$ if we insert $\mathcal{T}=\mathcal{F}$ and $\mathcal{S}=\mathcal{F}_N\mathcal{P}_N$ in to Theorem 4.1. Since $(\lambda-\mathcal{F})u=f$ is uniquely solvable we have $(\lambda-\mathcal{F})^{-1} \leq c_0 < \infty$. On the other hand $\|\mathcal{F}_N\mathcal{P}_N\| \leq c_P c_F$. Consequently, (13) implies

$$\|(\lambda-\mathcal{F}_N\mathcal{P}_N)^{-1}\| \leq \sup_{N\geq N_0} \frac{1+c_0c_Pc_F}{|\lambda|-c_0\|(\mathcal{F}-\mathcal{F}_N\mathcal{P}_N)\mathcal{F}_N\mathcal{P}_N\|} := c_I < \infty,$$

which proves the first assertion. If we set $w=u$ and $z=v_N$ in (14) then

$$\begin{aligned}
\|u-v_N\|_\infty &\leq \|(\lambda-\mathcal{F}_N\mathcal{P}_N)^{-1}\|\|\mathcal{F}u-\mathcal{F}_N\mathcal{P}_Nu\|_\infty \\
&\leq c_I\big\{\|\mathcal{F}u-\mathcal{F}_Nu\|_\infty+\|\mathcal{F}_N(u-\mathcal{P}_Nu)\|_\infty\big\} \\
&\leq c_I\big\{\|\mathcal{F}u-\mathcal{F}_Nu\|_\infty+c_F\|u-\mathcal{P}_Nu\|_\infty\big\} \\
&\leq c_I\big\{\|\mathcal{F}u-\mathcal{F}_Nu\|_\infty+c_F(\|u-\widehat{u}\|_\infty+\|\mathcal{P}_Nu-\mathcal{P}_N\widehat{u}\|_\infty)\big\} \\
&\leq c_I\big\{\|\mathcal{F}u-\mathcal{F}_Nu\|_\infty+c_F(1+c_P)\|u-\widehat{u}\|_\infty\big\} \\
&\leq c_I\big\{\|\mathcal{F}u-\mathcal{F}_Nu\|_\infty+c_F(1+c_P)Ch_{X,\Omega}^{m+1}\|u\|_{W_\infty^{m+1}(\Omega)}\big\}.
\end{aligned}$$

The last inequality is implied by (3). Moreover,

$$\begin{aligned}
u-u_N = u-\mathcal{P}_Nv_N &= (u-\mathcal{P}_Nu)+\mathcal{P}_N(u-v_N), \\
\|u-u_N\|_\infty &\leq \|u-\mathcal{P}_Nu\|_\infty + c_P\|u-v_N\|_\infty
\end{aligned}$$

lead to (17). □

Theorem 4.5 shows that, both the quadrature and the MLS approximation error bounds affect the final estimation. If for a sufficiently smooth kernel $\kappa(x,s)$ a high order quadrature is employed then the total error is dominated by the error of the MLS approximation. For numerical results see [2].

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